1 Infinite Series with $r = \frac{1}{2}$

Mr. Mikula buys a pizza for dinner. After eating half of the pizza, he continues to eat half of what is remaining. Still hungry, he eats half of what is left again. If he continues in this way, he will never eat the entire pizza, because there will always be half of a piece left.

a) Explain why this would not actually happen.

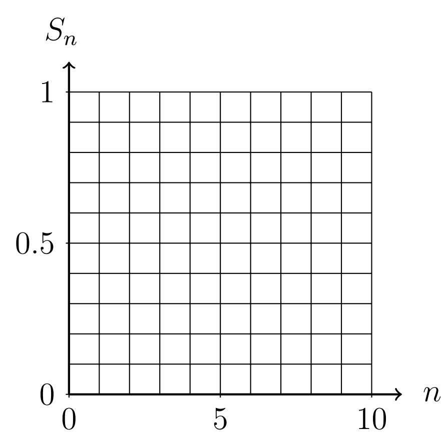
b) Represent this situation with a geometric sequence whose first term is $\frac{1}{2}$.

c) For this series, determine the values of a and r, and find a (simplified) formula for S_n.

d) Use a calculator to complete the table below to three decimal places:

n	1	2	3	4	5	6	7	8	9	10
S_n										

e) Plot the values from the table on the grid. Note that n is only defined on the set of natural numbers, so we **do not join the points**.



f) It would appear that as n gets larger, the sequence of the sums S_1, S_2, S_3, ... S_n gets closer and closer to the value _____.

This is called a **convergent** series, because the sequence of sums **converges** closer and closer to a certain value. It would appear that if we added the terms of the series forever, we would get closer and closer to 1.

We say that the sum of the infinite geometric series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...$ is 1, or that the sum to infinity of the series is 1.

We use the symbol S or S_∞ to represent the sum of the infinite series. Note that the sum of a *finite* number of terms of this series will never actually reach 1.

g) Explain (with reference to the pizza) why the sum of this infinite series should be 1.

2 Infinite series with $r = 2$

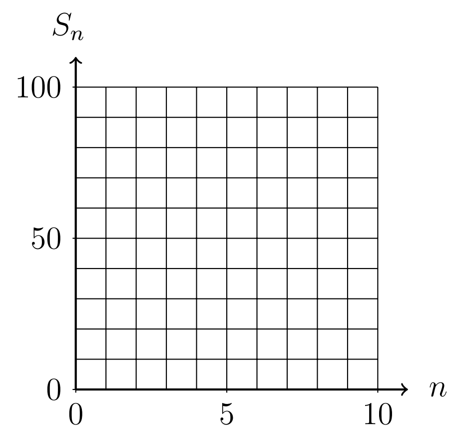
Consider the infinite geometric series $2 + 4 + 8 + 16 + ...$

a) For this series, determine the values of a and r, and find a (simplified) formula for S_n.

b) Use a calculator to complete the table below.

n	1	2	3	4	5	6	7	8	9	10
S_n										

c) Plot the values from the table on the grid.



In this case, as n gets larger, the sequence of sums S_1, S_2, S_3, ..., S_n gets larger and larger, and does not converge to a finite value.

This kind of series is called a **divergent** series, and does not have a sum to infinity.

So! An infinite geometric series with $r = \frac{1}{2}$ converges to a particular value (called the *sum to infinity*), and an infinite series with $r = 2$ diverges, and does not have a sum to infinity.

3 Values of r^n as n Approaches Infinity

In this investigation, we can choose:

- any value of r less than -1
- any value of r between -1 and 0
- any value of r between 0 and 1
- any value of r greater than 1.

a) Complete the table.

	$r < -1$	$-1 < r < 0$	$0 < r < 1$	$r > 1$
	$r =$ _____	$r =$ _____	$r =$ _____	$r =$ _____
n	r^n	r^n	r^n	r^n
1				
2				
3				
4				
10				
20				
100				

b) Complete the following statements based on your observations above.

As n gets larger and larger:

- the sequence is convergent and approaches the value 0 if $r =$ _____, or $r =$ _____.
- the sequence is divergent if $r =$ _____, or $r =$ _____.

As n gets larger and larger:

- r^n gets closer and closer to 0, provided that $-1 < r < 1$.
- r^n gets larger and larger if $r < -1$ or $r > 1$.

We can use this rule to determine a formula for the sum of an infinite geometric series.

4 The Formula for the Sum of an Infinite Geometric Series

Consider the infinite series $S = a + ar + ar^2 + ar^3 + ...$

The sum of n terms of this series is given by the formula $S_n = \frac{a(r^n-1)}{r-1}, r \neq 1$.

This formula can also be written as $S_n = \frac{a(1-r^n)}{1-r}, r \neq 1$.

As long as $-1 < r < 1$, r^n will get closer and closer to zero as n gets closer and closer to infinity. This means that in determining the formula for the sum of an infinite series, S, we can replace r^n with zero in the formula for S_n.

This gives us $S = \frac{a}{1-r}$, as long as $-1 < r < 1$.

If r is not between -1 and 1, the sum of an infinite geometric series is not defined.

$$\boxed{S = \frac{a}{1-r}, -1 < r < 1}$$

4.1 Example

Determine the common ratio for each of the following geometric series and state whether a sum to infinity exists. If it exists, calculate it.

a) $1 + \frac{1}{3} + \frac{1}{9} + ...$ b) $1 - 5 + 25 - ...$ c) $2 - 1 + \frac{1}{2} - ...$

4.2 Example

The first term of a geometric series is 2, and the sum of infinity is 4. Determine the common ratio.

4.3 Example

a) Write the repeating decimal $0.0\overline{7}$ as an infinite geometric series.

b) Find the sum to infinity of the series above, and write $0.0\overline{7}$ as a fraction.